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Let h be the largest altitude of a non obtuse triangle with circumradius R and inradius r. Show that $R+r \le h$.

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

Let $h = h_c = \max\{h_a,h_b,h_c\}$ that is $h_a,h_b \le h_c \Leftrightarrow \frac{1}{a},\frac{1}{b} \le \frac{1}{c} \Leftrightarrow C \le A,B.$

We have $R+r \le h = R+r \le \frac{ab}{2R} \Leftrightarrow 1+\frac{r}{R} \le \frac{ab}{2R^2} \Leftrightarrow$

$\cos A+\cos B+\cos C \le 2\sin A\sin B \Leftrightarrow \cos A+\cos B+\cos C \le \cos(A-B)-\cos(A+B) \Leftrightarrow$

$\cos A+\cos B+\cos C \le \cos(A-B)+\cos C \Leftrightarrow \cos A+\cos B \le \cos(A-B).$

Assume WLOG that $B \le A$. Then we obtain $\begin{cases} 0 < C \le B \le A \le \pi/2 \\ A+B+C = \pi \end{cases} \Leftrightarrow$

(**1**) $\begin{cases} \frac{\pi}{3} \le A \le \pi/2 \\ \frac{\pi-A}{2} \le B \le A \\ C = \pi - A - B \end{cases}.$

Since $\cos(A-B)$ and $-\cos B$ both increase by $B \in \left[\frac{\pi-A}{2},A\right]$ then

$\cos(A-B)-\cos B-\cos A \ge \cos\left(A-\frac{\pi-A}{2}\right)-\cos\frac{\pi-A}{2}-\cos A =$

$\sin\frac{3A}{2}-\sin\frac{A}{2}-\cos A = 2\cos A\cdot\sin\frac{A}{2}-\cos A = \cos A\left(2\sin\frac{A}{2}-1\right) \ge 0$

because $\cos A \ge 0$ and $2\sin\frac{A}{2} \ge 2\sin\frac{\pi}{4}-1 = \sqrt{2}-1 > 0.$

Since $\cos A\left(2\sin\frac{A}{2}-1\right) = 0 \Leftrightarrow A = \pi/2$ or $A = \pi/3$ and $B = \frac{\pi-A}{2}$

then equality in inequality $R+r \le h_c$ holds iff $(A,B,C) = (\pi/2,\pi/4,\pi/4)$ or $(A,B,C) = (\pi/3,\pi/3,\pi/3)$.